Week 5, problem 2.b notes

Laird Ware slides and ALA text make the very sepeciall assumption of equal variances at the two time points (pre, post). As shown in notes and Rogosa Brandt Zimowski (1982) on CD and links Eq 10 p.733 attached, this forces the correlation of change and initial status to be negative.

Calculations on following page, instead use the (annoying) week 2 handout results for collections of growth curves, to obtain the Laird-Ware results for the simple case of zero correlation between change and initial status.

Laird-Ware assertion (for their simplified calculation) slide 203. the residual variance of the analysis of covariance model is always smaller than the residual variance of the repeated measures (or change score) model

is shown not to hold

Artificial data examples could show more extreme cases

$\sigma_{\xi_1} = \sigma_{\xi_2}$ is specified,

$$\rho_{\xi_1\beta} = -\sqrt{\frac{1-\rho_{\xi_1\xi_2}}{2}}. \qquad (10)$$

Thus whenever a variable is standardized to have equal variance over time, the correlation between change and initial status must be less than or equal to zero. (An equivalent relation holds in terms of the X_i and D.) Both empirical and methodological investigations of change should heed the argument against standardization in Thorndike (1966): "By eliminating from the score scale the differences in standard deviation at different ages, that which is the essence of growth has been eliminated. . . . The constraint that has been put on the score scale assures distorted results" (p. 126).

In much of the literature on change the apparent dependence of $\rho(D)$ on $\rho_{X_1X_2}$ has been discussed as a "dilemma" or "paradox." The basis for the paradox is that when

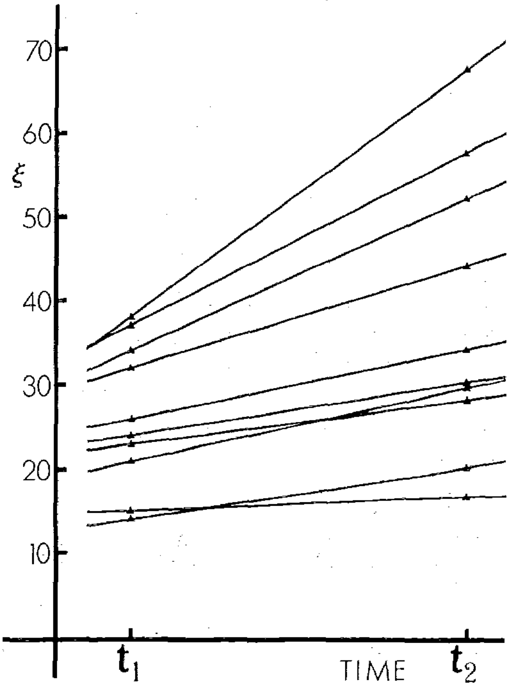

Figure 3. A configuration of individual time paths exhibiting individual differences in change and strong correlation between change and initial status. (For these time paths the correlation between ξ_1 and ξ_2 is .97 and the correlation between ξ_1 and β is .88.)

Table 3
Values of $\rho(D)/\rho(X)$ as a Function of $\rho_{\xi_1\xi_2}$ and $\rho(X)$

	$\rho(X)$		
$\rho_{\xi_1\xi_2}$	.7	.8	.9
.3	.886	.921	.959
.5	.769	.833	.909
.7	.588	.682	.811
.9	.270	.357	.526

$\rho(D)$ is displayed in the form shown in Table 2, $\rho(D)$ decreases as $\rho_{X_1X_2}$ increases (see in particular the Assumption III row of Table 2).[5] But stability has only an incidental role in understanding $\rho(D)$. To unravel the paradox it needs only to be recognized that both $\rho(D)$ and $\rho_{X_1X_2}$ depend on σ^2_β. This is documented by Equation 7 and Equation 8, respectively. (The inverse relation of σ^2_β and $\rho_{X_1X_2}$ is most clearly shown in Equation 9.) Selection of variables with high stability often results in small σ^2_β, and with little individual differences in true change to detect, $\rho(D)$ is small.

Furthermore, the major misconception that $\rho(D)$ is intrinsically small is a consequence of studying $\rho(D)$ only for very large $\rho_{\xi_1\xi_2}$. When X has high reliability and there exist individual differences to be detected, $\rho(D)$ will be respectable. The entries in Table 3 are values of the ratio $\rho(D)/\rho(X)$ calculated for different $\rho_{\xi_1\xi_2}$ (rows) and different $\rho(X)$ (columns).[6] Assumption III was used to sim-

[5] Using the logic that $\rho_{X_1X_2}$ will decrease as the time between observations is lengthened, writers on change have recommended that to increase $\rho(D)$ the time between observations should be made large. The dependence of $\rho(D)$ on $(t_2 - t_1)^2$ is shown in Equation 7. Extending the time between observations increases the precision of $\hat{\beta}_j$, and it is this increased precision that produces the increase in $\rho(D)$. The role of stability is indirect and secondary.

[6] Zimmerman, Brotohusodo, and Williams (1981) presented a detailed analysis of the discrepency $\rho(X) - \rho(D)$, with special attention to the effects of correlated errors on $\rho(D)$. Their analysis ignores the important role of σ^2_β and $\rho_{\xi_1\xi_2}$ in understanding $\rho(D)$, and curiously, the authors speak against the use of the difference score because $\rho(D)$ is difficult to estimate. Tables of $\rho(D)$ as a function of $\rho_{X_1X_2}$ and $\rho(X)$ in Kessler (1977, Table 1) and Stanley (1971, Table 13.2) represent the traditional approach to the study of $\rho(D)$.

w5 p2b

Ancova vs Gain Scores (random assignment)

ADA Laird-Ware Calculations

data perfectly measured $Y(t_2)$ $Y(t_1)$

Gain score $Y_2 - Y_1 = \Delta = (t_2 - t_1)\theta$ rate of change

$Var(Y_2 - Y_1) = Var(\Delta) = (t_2 - t_1)^2 \sigma_\theta^2$ compare w/ result for ancova

for special case $Cor(\Delta, Y_1) = 0$ $[t^\circ = t_1]$

From Week 2 handout (attached)

$Var(Y_1) = \sigma_1^2$ $Var(Y_2) = \sigma_1^2\left(1 + \left(\frac{t_2 - t_1}{K}\right)^2\right)$

for t° "t_1" $Cov(Y_1, Y_2) = \sigma_1^2$ $[K = \sigma_1/\sigma_\theta]$

$Cor(Y_1, Y_2) = \sigma_1/\sigma_2$ for $t^\circ = t_1$

then following Laird-Ware residual variance $\sigma_\Delta = (t_2 - t_1)\sigma_\theta$

$$Var(Y_2 \mid Y_1) = \sigma_2^2\left(1 - \frac{\sigma_1^2}{\sigma_2^2}\right) = \sigma_2^2 - \sigma_1^2$$

$$= \sigma_1^2\left(\frac{t_2 - t_1}{K}\right)^2 = \sigma_1^2\left(\frac{(t_2 - t_1)^2\sigma_\theta^2}{\sigma_1^2}\right) = Var(\Delta)$$

so ancova, gain score identical "variances" for $t_1 = t^\circ$ $(Cor(\Delta, Y_1) = 0)$

[L-W claim of equality for $Cor(Y_1, Y_2) = 1$ is that $\Rightarrow Cor(\Delta, Y_1) = 0$ RB2 eq 10]

note: Laird-Ware quantities aren't the complete calculation (no treatment effect etc)

Properties (Moments of Observables) of Collections of Growth Curves

STAT 222
Week 2
D Rogosa

for indiv p $\quad \xi_p(t) = \xi_p(0) + \theta_p t \qquad t_i \ (i = 1, \ldots T)$
$\qquad p \ (p = 1, \ldots n)$

centering, scale
$t^o = -\sigma_{\xi(0)\theta} / \sigma^2_\theta$
$\rho_{\xi(t^o)\theta} = 0$, min $Var(\xi)$
Scale $\kappa = \sigma_{\xi(t^o)} / \sigma_\theta$
time metric

"centering"

$$\xi_p(t) = \xi_p(t^o) + \theta_p(t - t^o)$$

Moments

Covariance $\sigma_{\xi(t_1)\xi(t_2)} =$
$$(t_1 - t^o)(t_2 - t^o)\sigma^2_\theta + \sigma^2_{\xi(t^o)}$$

Variance $\quad \sigma^2_{\xi(t)} = \sigma^2_{\xi(t^o)} + ((t - t^o)/\kappa)^2 \sigma^2_{\xi(t^o)}$
$$\sigma^2_{\xi(t)} / \sigma^2_{\xi(t^o)} = 1 + \left(\frac{t - t^o}{\kappa}\right)^2$$

Correl change, initial status $\quad \rho_{\theta\xi(t)} = \dfrac{t - t^o}{[\kappa^2 + (t - t^o)^2]^{1/2}}$

exogenous var W

$$\rho_{W\xi(t)} = \frac{(t - t^o)\rho_{W\theta} + \kappa\rho_{W\xi(t^o)}}{[\kappa^2 + (t - t^o)^2]^{1/2}}$$

where $\quad t^u = t^o + \kappa\left(\dfrac{\rho_{W\theta}}{\rho_{W\xi(t^o)}}\right) \qquad t^l = t^o - \kappa\left(\dfrac{\rho_{W\xi(t^o)}}{\rho_{W\theta}}\right)$

Week 1 example: (p.64) $\theta \sim U[1, 9]$, $\xi(t^o) \sim U[38, 62]$
$t^o = 2 \qquad \sigma^2_\theta = 5.333 \quad \sigma^2_{\xi(t^o)} = 48 \qquad \rho_{W\theta} = 0 \quad \rho_{W\xi(t^o)} =$

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at time t_i $\quad X_{ip} = \xi_{ip} + \epsilon \qquad \epsilon \sim (0, \sigma^2_\epsilon) \qquad$ errors in variables
week 1 ex $\sigma^2_\epsilon = 10$