```
<< mathStatica.m
```

■ mathStatica :	v1.0	■ Registered to :	Prof David Rogosa
■ No. of licenses :	2	■ User class :	Gold / Professional

```
(* Nonparametric Delta Method for Variance of Correlation Coefficient
    Efron & Tibshirani section 21.9 result on page 315  *)

(* for bivariate data (X,Y), n observations, form the vector
     Q = {X, Y, X^2, XY, Y^2}  and for q[[i]] = mean (Q[[i]])
   the sample correlation coefficient is given by rfunc below *)

rfunc[q1_, q2_, q3_, q4_, q5_] := (q4 - q1 * q2) /
                                    Sqrt[(q3 - q1^2) * (q5 - q2^2)]

(* Step 1: Obtain ∇r in E& F equation 21.67  with the del function below *)

ident = IdentityMatrix[5];

delCorr[q1_, q2_, q3_, q4_, q5_] :=
 Table[FullSimplify[Derivative[ident[[i, 1]], ident[[i, 2]], ident[[i, 3]],
      ident[[i, 4]], ident[[i, 5]]][rfunc][q1, q2, q3, q4, q5]], {i, 1, 5}]

delCorr[q1, q2, q3, q4, q5]
```

$$\left\{\frac{(q2\,q3-q1\,q4)\,(q2^2-q5)}{((q1^2-q3)\,(q2^2-q5))^{3/2}},\ \frac{(q1^2-q3)\,(-q2\,q4+q1\,q5)}{((q1^2-q3)\,(q2^2-q5))^{3/2}},\ -\frac{(-q1\,q2+q4)\,(-q2^2+q5)}{2\,((q1^2-q3)\,(q2^2-q5))^{3/2}},\ \frac{1}{\sqrt{(q1^2-q3)\,(q2^2-q5)}},\ -\frac{(-q1^2+q3)\,(-q1\,q2+q4)}{2\,((q1^2-q3)\,(q2^2-q5))^{3/2}}\right\}$$

```
(* that's a useful bit of calculus done for us instantaneously;
 because the final result, eq 21.70, is expressed in central moments,
 it's useful to reformat the result above in order to convert from
  raw moments to central moments *)
```

delCorr$[\acute{\mu}_{1,0},\ \acute{\mu}_{0,1},\ \acute{\mu}_{2,0},\ \acute{\mu}_{1,1},\ \acute{\mu}_{0,2}]$

$$\left\{\frac{(\acute{\mu}_{0,1}^2-\acute{\mu}_{0,2})\,(-\acute{\mu}_{1,0}\,\acute{\mu}_{1,1}+\acute{\mu}_{0,1}\,\acute{\mu}_{2,0})}{\left((\acute{\mu}_{0,1}^2-\acute{\mu}_{0,2})\,(\acute{\mu}_{1,0}^2-\acute{\mu}_{2,0})\right)^{3/2}},\ \frac{(\acute{\mu}_{0,2}\,\acute{\mu}_{1,0}-\acute{\mu}_{0,1}\,\acute{\mu}_{1,1})\,(\acute{\mu}_{1,0}^2-\acute{\mu}_{2,0})}{\left((\acute{\mu}_{0,1}^2-\acute{\mu}_{0,2})\,(\acute{\mu}_{1,0}^2-\acute{\mu}_{2,0})\right)^{3/2}},\ -\frac{(-\acute{\mu}_{0,1}^2+\acute{\mu}_{0,2})\,(-\acute{\mu}_{0,1}\,\acute{\mu}_{1,0}+\acute{\mu}_{1,1})}{2\left((\acute{\mu}_{0,1}^2-\acute{\mu}_{0,2})\,(\acute{\mu}_{1,0}^2-\acute{\mu}_{2,0})\right)^{3/2}},\ \frac{1}{\sqrt{(\acute{\mu}_{0,1}^2-\acute{\mu}_{0,2})\,(\acute{\mu}_{1,0}^2-\acute{\mu}_{2,0})}},\ -\frac{(-\acute{\mu}_{0,1}\,\acute{\mu}_{1,0}+\acute{\mu}_{1,1})\,(-\acute{\mu}_{1,0}^2+\acute{\mu}_{2,0})}{2\left((\acute{\mu}_{0,1}^2-\acute{\mu}_{0,2})\,(\acute{\mu}_{1,0}^2-\acute{\mu}_{2,0})\right)^{3/2}}\right\}$$

```
(* transformation from raw to central moments is a two-step
    procedure, using the facilities in mathStatica *)

? RawToCumulant
```

RawToCumulant[r] expresses the rth raw moment $\acute{\mu}_r$ in terms of cumulants κ_i. To obtain a multivariate conversion, let r be a list of integers.

```
? CumulantToCentral
```

CumulantToCentral[r] expresses the rth cumulant κ_r in terms of central moments μ_i. To obtain a multivariate conversion, let r be a list of integers.

```
rulesDelRawtoK = {RawToCumulant[{0, 1}], RawToCumulant[{0, 2}],
  RawToCumulant[{1, 0}], RawToCumulant[{1, 1}], RawToCumulant[{2, 0}]}
```

$$\{\acute{\mu}_{0,1}\to\kappa_{0,1},\ \acute{\mu}_{0,2}\to\kappa_{0,1}^2+\kappa_{0,2},\ \acute{\mu}_{1,0}\to\kappa_{1,0},\ \acute{\mu}_{1,1}\to\kappa_{0,1}\,\kappa_{1,0}+\kappa_{1,1},\ \acute{\mu}_{2,0}\to\kappa_{1,0}^2+\kappa_{2,0}\}$$

delCorrinK = delCorr[$\acute{\mu}_{1,0}$ **,** $\acute{\mu}_{0,1}$ **,** $\acute{\mu}_{2,0}$ **,** $\acute{\mu}_{1,1}$ **,** $\acute{\mu}_{0,2}$ **] /. rulesDelRawtoK**

$$\left\{-\frac{\kappa_{0,2}\,(-\kappa_{1,0}\,(\kappa_{0,1}\,\kappa_{1,0}+\kappa_{1,1})+\kappa_{0,1}\,(\kappa_{1,0}^2+\kappa_{2,0}))}{(\kappa_{0,2}\,\kappa_{2,0})^{3/2}},\ -\frac{((\kappa_{0,1}^2+\kappa_{0,2})\,\kappa_{1,0}-\kappa_{0,1}\,(\kappa_{0,1}\,\kappa_{1,0}+\kappa_{1,1}))\,\kappa_{2,0}}{(\kappa_{0,2}\,\kappa_{2,0})^{3/2}},\ -\frac{\kappa_{0,2}\,\kappa_{1,1}}{2\,(\kappa_{0,2}\,\kappa_{2,0})^{3/2}},\ \frac{1}{\sqrt{\kappa_{0,2}\,\kappa_{2,0}}},\ -\frac{\kappa_{1,1}\,\kappa_{2,0}}{2\,(\kappa_{0,2}\,\kappa_{2,0})^{3/2}}\right\}$$

FullSimplify[%]

$$\left\{\frac{\kappa_{0,2}\,(\kappa_{1,0}\,\kappa_{1,1}-\kappa_{0,1}\,\kappa_{2,0})}{(\kappa_{0,2}\,\kappa_{2,0})^{3/2}},\ \frac{(-\kappa_{0,2}\,\kappa_{1,0}+\kappa_{0,1}\,\kappa_{1,1})\,\kappa_{2,0}}{(\kappa_{0,2}\,\kappa_{2,0})^{3/2}},\ -\frac{\kappa_{0,2}\,\kappa_{1,1}}{2\,(\kappa_{0,2}\,\kappa_{2,0})^{3/2}},\ \frac{1}{\sqrt{\kappa_{0,2}\,\kappa_{2,0}}},\ -\frac{\kappa_{1,1}\,\kappa_{2,0}}{2\,(\kappa_{0,2}\,\kappa_{2,0})^{3/2}}\right\}$$

```
rulesDelKtoC = {CumulantToRaw[{1, 0}], CumulantToRaw[{0, 1}],
  CumulantToCentral[{2, 0}], CumulantToCentral[{1, 1}], CumulantToCentral[{0, 2}]}
```

$$\left\{\kappa_{1,0}\to\acute{\mu}_{1,0},\ \kappa_{0,1}\to\acute{\mu}_{0,1},\ \kappa_{2,0}\to\mu_{2,0},\ \kappa_{1,1}\to\mu_{1,1},\ \kappa_{0,2}\to\mu_{0,2}\right\}$$

```
delCorrinC = FullSimplify[delCorrinK /. rulesDelKtoC]
```

$$\left\{\frac{\mu_{0,2}\,\left(-\mu_{2,0}\,\acute{\mu}_{0,1}+\mu_{1,1}\,\acute{\mu}_{1,0}\right)}{(\mu_{0,2}\,\mu_{2,0})^{3/2}},\ \frac{\mu_{2,0}\,\left(\mu_{1,1}\,\acute{\mu}_{0,1}-\mu_{0,2}\,\acute{\mu}_{1,0}\right)}{(\mu_{0,2}\,\mu_{2,0})^{3/2}},\ -\frac{\mu_{0,2}\,\mu_{1,1}}{2\,(\mu_{0,2}\,\mu_{2,0})^{3/2}},\ \frac{1}{\sqrt{\mu_{0,2}\,\mu_{2,0}}},\ -\frac{\mu_{1,1}\,\mu_{2,0}}{2\,(\mu_{0,2}\,\mu_{2,0})^{3/2}}\right\}$$

```
(* Step 2: Build up the 5x5 covariance matrix of the Q vector
    I do so element by element for simple exposition; there are
    more elegant forms for this *)
```

sig11 = $\mu_{2,0}$ **; sig22 =** $\mu_{0,2}$ **; sig12 = sig21 =** $\mu_{1,1}$ **;**

```
sig33 = \!\(\(-μ \&′\_\(2, 0\) \%2\) + \μ \&′\_\(4, 0\)\);
sig55 = \!\(\(-μ \&′\_\(0, 2\) \%2\) + \μ \&′\_\(0, 4\)\);
sig44 = \!\(\(-μ \&′\_\(1, 1\) \%2\) + \μ \&′\_\(2, 2\)\);
```

sig13 = sig31 = $-\acute{\mu}_{1,0}\,\acute{\mu}_{2,0}+\acute{\mu}_{3,0}$**; sig25 = sig52 =** $-\acute{\mu}_{0,1}\,\acute{\mu}_{0,2}+\acute{\mu}_{0,3}$**;**

sig14 = sig41 = $-\acute{\mu}_{1,1}\,\acute{\mu}_{1,0}+\acute{\mu}_{2,1}$ **;**

sig15 = sig51 = $-\acute{\mu}_{0,2}\,\acute{\mu}_{1,0}+\acute{\mu}_{1,2}$ **; sig23 = sig32 =** $-\acute{\mu}_{2,0}\,\acute{\mu}_{0,1}+\acute{\mu}_{2,1}$ **;**

sig24 = sig42 = $-\acute{\mu}_{1,1}\,\acute{\mu}_{0,1}+\acute{\mu}_{1,2}$ **; sig34 = sig43 =** $-\acute{\mu}_{1,1}\,\acute{\mu}_{2,0}+\acute{\mu}_{3,1}$ **;**

sig35 = sig53 = $-\acute{\mu}_{0,2}\,\acute{\mu}_{2,0}+\acute{\mu}_{2,2}$ **; sig45 = sig54 =** $-\acute{\mu}_{1,1}\,\acute{\mu}_{0,2}+\acute{\mu}_{1,3}$**;**

```
sigRaw = {{sig11, sig12, sig13, sig14, sig15},
  {sig21, sig22, sig23, sig24, sig25}, {sig31, sig32, sig33, sig34, sig35},
  {sig41, sig42, sig43, sig44, sig45}, {sig51, sig52, sig53, sig54, sig55}}
```

$$\begin{pmatrix} \mu_{2,0} & \mu_{1,1} & -\acute{\mu}_{1,0}\,\acute{\mu}_{2,0}+\acute{\mu}_{3,0} & -\acute{\mu}_{1,0}\,\acute{\mu}_{1,1}+\acute{\mu}_{2,1} & -\acute{\mu}_{0,2}\,\acute{\mu}_{1,0}+\acute{\mu}_{1,2} \\ \mu_{1,1} & \mu_{0,2} & -\acute{\mu}_{0,1}\,\acute{\mu}_{2,0}+\acute{\mu}_{2,1} & -\acute{\mu}_{0,1}\,\acute{\mu}_{1,1}+\acute{\mu}_{1,2} & -\acute{\mu}_{0,1}\,\acute{\mu}_{0,2}+\acute{\mu}_{0,3} \\ -\acute{\mu}_{1,0}\,\acute{\mu}_{2,0}+\acute{\mu}_{3,0} & -\acute{\mu}_{0,1}\,\acute{\mu}_{2,0}+\acute{\mu}_{2,1} & -\acute{\mu}_{2,0}^{2}+\acute{\mu}_{4,0} & -\acute{\mu}_{1,1}\,\acute{\mu}_{2,0}+\acute{\mu}_{3,1} & -\acute{\mu}_{0,2}\,\acute{\mu}_{2,0}+\acute{\mu}_{2,2} \\ -\acute{\mu}_{1,0}\,\acute{\mu}_{1,1}+\acute{\mu}_{2,1} & -\acute{\mu}_{0,1}\,\acute{\mu}_{1,1}+\acute{\mu}_{1,2} & -\acute{\mu}_{1,1}\,\acute{\mu}_{2,0}+\acute{\mu}_{3,1} & -\acute{\mu}_{1,1}^{2}+\acute{\mu}_{2,2} & -\acute{\mu}_{0,2}\,\acute{\mu}_{1,1}+\acute{\mu}_{1,3} \\ -\acute{\mu}_{0,2}\,\acute{\mu}_{1,0}+\acute{\mu}_{1,2} & -\acute{\mu}_{0,1}\,\acute{\mu}_{0,2}+\acute{\mu}_{0,3} & -\acute{\mu}_{0,2}\,\acute{\mu}_{2,0}+\acute{\mu}_{2,2} & -\acute{\mu}_{0,2}\,\acute{\mu}_{1,1}+\acute{\mu}_{1,3} & -\acute{\mu}_{0,2}^{2}+\acute{\mu}_{0,4} \end{pmatrix}$$

FullSimplify[%]

$$\begin{pmatrix} \mu_{2,0} & \mu_{1,1} & -\acute{\mu}_{1,0}\,\acute{\mu}_{2,0}+\acute{\mu}_{3,0} & -\acute{\mu}_{1,0}\,\acute{\mu}_{1,1}+\acute{\mu}_{2,1} & -\acute{\mu}_{0,2}\,\acute{\mu}_{1,0}+\acute{\mu}_{1,2} \\ \mu_{1,1} & \mu_{0,2} & -\acute{\mu}_{0,1}\,\acute{\mu}_{2,0}+\acute{\mu}_{2,1} & -\acute{\mu}_{0,1}\,\acute{\mu}_{1,1}+\acute{\mu}_{1,2} & -\acute{\mu}_{0,1}\,\acute{\mu}_{0,2}+\acute{\mu}_{0,3} \\ -\acute{\mu}_{1,0}\,\acute{\mu}_{2,0}+\acute{\mu}_{3,0} & -\acute{\mu}_{0,1}\,\acute{\mu}_{2,0}+\acute{\mu}_{2,1} & -\acute{\mu}_{2,0}^{2}+\acute{\mu}_{4,0} & -\acute{\mu}_{1,1}\,\acute{\mu}_{2,0}+\acute{\mu}_{3,1} & -\acute{\mu}_{0,2}\,\acute{\mu}_{2,0}+\acute{\mu}_{2,2} \\ -\acute{\mu}_{1,0}\,\acute{\mu}_{1,1}+\acute{\mu}_{2,1} & -\acute{\mu}_{0,1}\,\acute{\mu}_{1,1}+\acute{\mu}_{1,2} & -\acute{\mu}_{1,1}\,\acute{\mu}_{2,0}+\acute{\mu}_{3,1} & -\acute{\mu}_{1,1}^{2}+\acute{\mu}_{2,2} & -\acute{\mu}_{0,2}\,\acute{\mu}_{1,1}+\acute{\mu}_{1,3} \\ -\acute{\mu}_{0,2}\,\acute{\mu}_{1,0}+\acute{\mu}_{1,2} & -\acute{\mu}_{0,1}\,\acute{\mu}_{0,2}+\acute{\mu}_{0,3} & -\acute{\mu}_{0,2}\,\acute{\mu}_{2,0}+\acute{\mu}_{2,2} & -\acute{\mu}_{0,2}\,\acute{\mu}_{1,1}+\acute{\mu}_{1,3} & -\acute{\mu}_{0,2}^{2}+\acute{\mu}_{0,4} \end{pmatrix}$$

```
rulesSigmaRawtoK = Flatten[Table[RawToCumulant[{i, j}], {i, 0, 4}, {j, 0, 4}]];

sigK = FullSimplify[sigRaw /. rulesSigmaRawtoK];
```

```
rulesSigmaKtoC = {CumulantToRaw[{1, 0}], CumulantToRaw[{0, 1}],
  CumulantToCentral[{2, 0}], CumulantToCentral[{1, 1}],
  CumulantToCentral[{0, 2}], CumulantToCentral[{3, 0}],
  CumulantToCentral[{0, 3}], CumulantToCentral[{2, 1}],
  CumulantToCentral[{1, 2}], CumulantToCentral[{2, 2}], CumulantToCentral[{3, 1}],
  CumulantToCentral[{4, 0}], CumulantToCentral[{0, 4}], CumulantToCentral[{1, 3}],
  CumulantToCentral[{2, 3}], CumulantToCentral[{4, 1}],
  CumulantToCentral[{3, 2}], CumulantToCentral[{1, 4}]}
```

$\{\kappa_{1,0} \to \acute{\mu}_{1,0},\ \kappa_{0,1} \to \acute{\mu}_{0,1},\ \kappa_{2,0} \to \mu_{2,0},\ \kappa_{1,1} \to \mu_{1,1},\ \kappa_{0,2} \to \mu_{0,2},\ \kappa_{3,0} \to \mu_{3,0},\ \kappa_{0,3} \to \mu_{0,3},$
$\kappa_{2,1} \to \mu_{2,1},\ \kappa_{1,2} \to \mu_{1,2},\ \kappa_{2,2} \to -2\,\mu_{1,1}^2 - \mu_{0,2}\,\mu_{2,0} + \mu_{2,2},\ \kappa_{3,1} \to -3\,\mu_{1,1}\,\mu_{2,0} + \mu_{3,1},$
$\kappa_{4,0} \to -3\,\mu_{2,0}^2 + \mu_{4,0},\ \kappa_{0,4} \to -3\,\mu_{0,2}^2 + \mu_{0,4},\ \kappa_{1,3} \to -3\,\mu_{0,2}\,\mu_{1,1} + \mu_{1,3},$
$\kappa_{2,3} \to -6\,\mu_{1,1}\,\mu_{1,2} - \mu_{0,3}\,\mu_{2,0} - 3\,\mu_{0,2}\,\mu_{2,1} + \mu_{2,3},\ \kappa_{4,1} \to -6\,\mu_{2,0}\,\mu_{2,1} - 4\,\mu_{1,1}\,\mu_{3,0} + \mu_{4,1},$
$\kappa_{3,2} \to -3\,\mu_{1,2}\,\mu_{2,0} - 6\,\mu_{1,1}\,\mu_{2,1} - \mu_{0,2}\,\mu_{3,0} + \mu_{3,2},\ \kappa_{1,4} \to -4\,\mu_{0,3}\,\mu_{1,1} - 6\,\mu_{0,2}\,\mu_{1,2} + \mu_{1,4}\}$

```
sigCentral = FullSimplify[sigK /. rulesSigmaKtoC];
```

```
TableForm[sigCentral[[1]]]
```

$\mu_{2,0}$
$\mu_{1,1}$
$\mu_{3,0} + 2\,\mu_{2,0}\,\acute{\mu}_{1,0}$
$\mu_{2,1} + \mu_{2,0}\,\acute{\mu}_{0,1} + \mu_{1,1}\,\acute{\mu}_{1,0}$
$\mu_{1,2} + 2\,\mu_{1,1}\,\acute{\mu}_{0,1}$

```
TableForm[sigCentral[[2]]]
```

$\mu_{1,1}$
$\mu_{0,2}$
$\mu_{2,1} + 2\,\mu_{1,1}\,\acute{\mu}_{1,0}$
$\mu_{1,2} + \mu_{1,1}\,\acute{\mu}_{0,1} + \mu_{0,2}\,\acute{\mu}_{1,0}$
$\mu_{0,3} + 2\,\mu_{0,2}\,\acute{\mu}_{0,1}$

```
TableForm[sigCentral[[3]]]
```

$\mu_{3,0} + 2\,\mu_{2,0}\,\acute{\mu}_{1,0}$
$\mu_{2,1} + 2\,\mu_{1,1}\,\acute{\mu}_{1,0}$
$-\mu_{2,0}^2 + \mu_{4,0} + 4\,\mu_{3,0}\,\acute{\mu}_{1,0} + 4\,\mu_{2,0}\,\acute{\mu}_{1,0}^2$
$\mu_{3,1} + 3\,\mu_{2,1}\,\acute{\mu}_{1,0} + \acute{\mu}_{0,1}\,(\mu_{3,0} + 2\,\mu_{2,0}\,\acute{\mu}_{1,0}) - \mu_{1,1}\,(\mu_{2,0} - 2\,\acute{\mu}_{1,0}^2)$
$-\mu_{0,2}\,\mu_{2,0} + \mu_{2,2} + 2\,\mu_{2,1}\,\acute{\mu}_{0,1} + 2\,(\mu_{1,2} + 2\,\mu_{1,1}\,\acute{\mu}_{0,1})\,\acute{\mu}_{1,0}$

```
TableForm[sigCentral[[4]]]
```

$\mu_{2,1} + \mu_{2,0}\,\acute{\mu}_{0,1} + \mu_{1,1}\,\acute{\mu}_{1,0}$
$\mu_{1,2} + \mu_{1,1}\,\acute{\mu}_{0,1} + \mu_{0,2}\,\acute{\mu}_{1,0}$
$\mu_{3,1} + 3\,\mu_{2,1}\,\acute{\mu}_{1,0} + \acute{\mu}_{0,1}\,(\mu_{3,0} + 2\,\mu_{2,0}\,\acute{\mu}_{1,0}) - \mu_{1,1}\,(\mu_{2,0} - 2\,\acute{\mu}_{1,0}^2)$
$-\mu_{1,1}^2 + \mu_{2,2} + 2\,\mu_{2,1}\,\acute{\mu}_{0,1} + \mu_{2,0}\,\acute{\mu}_{0,1}^2 + 2\,(\mu_{1,2} + \mu_{1,1}\,\acute{\mu}_{0,1})\,\acute{\mu}_{1,0} + \mu_{0,2}\,\acute{\mu}_{1,0}^2$
$-\mu_{0,2}\,\mu_{1,1} + \mu_{1,3} + 3\,\mu_{1,2}\,\acute{\mu}_{0,1} + 2\,\mu_{1,1}\,\acute{\mu}_{0,1}^2 + (\mu_{0,3} + 2\,\mu_{0,2}\,\acute{\mu}_{0,1})\,\acute{\mu}_{1,0}$

```
TableForm[sigCentral[[5]]]
```

$\mu_{1,2} + 2\,\mu_{1,1}\,\acute{\mu}_{0,1}$
$\mu_{0,3} + 2\,\mu_{0,2}\,\acute{\mu}_{0,1}$
$-\mu_{0,2}\,\mu_{2,0} + \mu_{2,2} + 2\,\mu_{2,1}\,\acute{\mu}_{0,1} + 2\,(\mu_{1,2} + 2\,\mu_{1,1}\,\acute{\mu}_{0,1})\,\acute{\mu}_{1,0}$
$-\mu_{0,2}\,\mu_{1,1} + \mu_{1,3} + 3\,\mu_{1,2}\,\acute{\mu}_{0,1} + 2\,\mu_{1,1}\,\acute{\mu}_{0,1}^2 + (\mu_{0,3} + 2\,\mu_{0,2}\,\acute{\mu}_{0,1})\,\acute{\mu}_{1,0}$
$-\mu_{0,2}^2 + \mu_{0,4} + 4\,\mu_{0,3}\,\acute{\mu}_{0,1} + 4\,\mu_{0,2}\,\acute{\mu}_{0,1}^2$

```
Dimensions[sigCentral]
```

{5, 5}

```
(* Step 3: Now form vetor-matrix product ∇' Σ ∇ E& F eq 21.67 *)
```

```
result = delCorrinC.sigCentral.delCorrinC // FullSimplify
```

$$\frac{1}{4\,\mu_{0,2}^3\,\mu_{2,0}^3}\left(\mu_{0,4}\,\mu_{1,1}^2\,\mu_{2,0}^2+\mu_{0,2}\left(4\,\mu_{0,2}\,\mu_{2,0}^2\,\mu_{2,2}-4\,\mu_{1,1}\,\mu_{2,0}\left(\mu_{1,3}\,\mu_{2,0}+\mu_{0,2}\,\mu_{3,1}\right)+\mu_{1,1}^2\left(2\,\mu_{2,0}\,\mu_{2,2}+\mu_{0,2}\,\mu_{4,0}\right)\right)\right)$$

```
Expand[result]
```

$$\frac{\mu_{0,4}\,\mu_{1,1}^2}{4\,\mu_{0,2}^3\,\mu_{2,0}}-\frac{\mu_{1,1}\,\mu_{1,3}}{\mu_{0,2}^2\,\mu_{2,0}}+\frac{\mu_{1,1}^2\,\mu_{2,2}}{2\,\mu_{0,2}^2\,\mu_{2,0}^2}+\frac{\mu_{2,2}}{\mu_{0,2}\,\mu_{2,0}}-\frac{\mu_{1,1}\,\mu_{3,1}}{\mu_{0,2}\,\mu_{2,0}^2}+\frac{\mu_{1,1}^2\,\mu_{4,0}}{4\,\mu_{0,2}\,\mu_{2,0}^3}$$

```
(* this is n*result in E& F eq 21.70; to match the algebraic
    form need to pull out the factor corr^2/4 *)
```

```
Expand[result * 4 / (μ²₁,₁ / (μ₀,₂ μ₂,₀))]
```

$$\frac{\mu_{0,4}}{\mu_{0,2}^2}-\frac{4\,\mu_{1,3}}{\mu_{0,2}\,\mu_{1,1}}+\frac{4\,\mu_{2,2}}{\mu_{1,1}^2}+\frac{2\,\mu_{2,2}}{\mu_{0,2}\,\mu_{2,0}}-\frac{4\,\mu_{3,1}}{\mu_{1,1}\,\mu_{2,0}}+\frac{\mu_{4,0}}{\mu_{2,0}^2}$$

```
(* the above gives the terms within the brackets in eq 21.70 *)
```

Textbook pages attached

21.9 The delta method

The delta method is a special technique for variance estimation that is applicable to statistics that are functions of observed averages. Suppose that we can write

$$\hat{\theta}(X_1, X_2, \ldots X_n) = r(\bar{Q}_1, \bar{Q}_2, \ldots \bar{Q}_A), \tag{21.59}$$

where $r(\cdot, \cdot, \ldots \cdot)$ is a known function and

$$\bar{Q}_a = \frac{1}{n} \sum_1^n Q_a(X_i). \tag{21.60}$$

The simplest example is the mean, for which $Q_a(X_i) = X_i$; for the correlation we take

$$r(\bar{Q}_1, \bar{Q}_2, \bar{Q}_3, \bar{Q}_4, \bar{Q}_5) = \frac{\bar{Q}_4 - \bar{Q}_1\bar{Q}_2}{[\bar{Q}_3 - \bar{Q}_1^2]^{1/2}[\bar{Q}_5 - \bar{Q}_2^2]^{1/2}} \tag{21.61}$$

with $X = (Y, Z), Q_1(X) = Y, Q_2(X) = Z, Q_3(X) = Y^2, Q_4(X) = YZ, Q_5(X) = Z^2$.

The idea behind the delta method is the following. Suppose we have a random variable U with mean μ and variance σ^2, and we seek the variance of a one-to-one function of U, say $g(U)$. By expanding $g(t)$ in a one term Taylor series about $t = \mu$ we have

$$g(t) \approx g(\mu) + (t - \mu)g'(\mu) \tag{21.62}$$

and this gives

$$\mathrm{var}(g(U)) \approx [g'(\mu)]^2\sigma^2. \tag{21.63}$$

Now if U itself is a sample mean so that estimates of its mean μ and variance σ^2 are readily available, we can use this formula to obtain a simple estimate of $\mathrm{var}(g(U))$.

The delta method uses a multivariate version of this argument. Suppose $(Q_1(X), \dots Q_A(X))$ has mean vector $\boldsymbol{\mu}_F$. A multivariate Taylor series has the form

$$r(q_1, q_2 \dots q_A) \approx r(\mu_1, \mu_2, \dots \mu_A) + \sum_1^A (q_i - \mu_u)\frac{\partial r}{\partial q_i}|_{q_i=\mu_i}, \tag{21.64}$$

or in convenient vector notation

$$r(\mathbf{q}) \approx r(\boldsymbol{\mu}) + \nabla r^T(\mathbf{q} - \boldsymbol{\mu}). \tag{21.65}$$

This gives

$$\mathrm{var}(r(\bar{\mathbf{Q}})) \approx \frac{\nabla r_F^T \boldsymbol{\Sigma}_F \nabla r_F}{n} \tag{21.66}$$

where $\boldsymbol{\Sigma}_F$ is the variance-covariance matrix of a single observation $X \sim F$. We have put the subscript F on the quantities in (21.66) to remind ourselves that they depend on the unknown distribution F. The *nonparametric delta method* substitutes $\hat{F}$ for F in (21.66); this simply entails estimation of the first and second moments of X by their sample (plug-in) estimates:

$$\mathrm{var}^{ND}(r(\bar{\mathbf{Q}})) = \frac{\nabla r_{\hat{F}}^T \boldsymbol{\Sigma}_{\hat{F}} \nabla r_{\hat{F}}}{n}. \tag{21.67}$$

The *parametric delta method* uses a parametric estimate $F_{\hat{\eta}}$ for F:

$$\mathrm{var}^{PD}(r(\bar{\mathbf{q}})) = \frac{\nabla r_{\hat{\eta}}^T \boldsymbol{\Sigma}_{F_{\hat{\eta}}} \nabla r_{\hat{\eta}}}{n}. \tag{21.68}$$

In both the nonparametric and parametric versions, the fact that the statistic $\hat{\theta}$ is a function of sample means is the key aspect of the delta method.

21.9.1 *Example: delta method for the mean*

Here $Q_1(X) = X$, $r(q) = q$, $\nabla r_F = 1$, $\boldsymbol{\Sigma}_F = \mathrm{var}(X)$. The nonparametric delta method gives $\boldsymbol{\Sigma}_{\hat{F}} = \sum_1^n (x_i - \bar{x})^2/n$, the plug-in estimate of variance and finally

$$\mathrm{var}^{ND}\bar{X} = \sum_1^n (x_i - \bar{x})^2/n^2, \tag{21.69}$$

which equals the bootstrap or plug-in estimate of variance of the mean.

If we use a parametric estimate $F_{\hat{\eta}}$ then parametric delta method estimate is $\sigma^2(F_{\hat{\eta}})/n$. For example, if we assume X has an exponential distribution (21.31), then $\hat{\theta} = \bar{x}$, $\sigma^2(F_{\hat{\eta}}) = 1/\bar{x}^2$ and $\mathrm{var}^{PD}(\bar{X}) = 1/(n\bar{x}^2)$.

21.9.2 *Example: delta method for the correlation coefficient*

Application of the delta method to the correlation coefficient (21.61) shows how quickly the calculations can get complicated. Here $X = (Y, Z), Q_1(X) = Y, Q_2(X) = Z, Q_3(X) = Y^2, Q_4(X) = YZ$, $Q_5(X) = Z^2$. Letting $\beta_{ab} = \mathrm{E}_F[(Y - \mathrm{E}_F Y)^a(Z - \mathrm{E}_F Z)^b]$, after a long calculation, (21.67) gives

$$\mathrm{var}^{ND} r(\bar{\mathbf{Q}}) = \frac{\hat{\theta}^2}{4n}\Big[\frac{\hat{\beta}_{40}}{\hat{\beta}_{20}^2} + \frac{\hat{\beta}_{04}}{\hat{\beta}_{02}^2} + \frac{2\hat{\beta}_{22}}{\hat{\beta}_{20}\hat{\beta}_{02}} + \frac{4\hat{\beta}_{22}}{\hat{\beta}_{11}^2} - \frac{4\hat{\beta}_{31}}{\hat{\beta}_{11}\hat{\beta}_{20}} - \frac{4\hat{\beta}_{13}}{\hat{\beta}_{11}\hat{\beta}_{02}}\Big] \tag{21.70}$$

where each $\hat{\beta}_{ab}$ is a (plug-in) sample moment, for example $\hat{\beta}_{13} = \sum(y_i - \bar{y})(z_i - \bar{z})^3/n$. The parametric delta method would use a (bivariate) parametric estimate $F_{\hat{\eta}}$ and then $\hat{\beta}_{ab}$ would be the estimated moments from $F_{\hat{\eta}}$.

21.10 Relationship between the delta method and infinitesimal jackknife

The infinitesimal jackknife applies to general functional statistics while the delta method works only for functions of means. Interestingly, when the infinitesimal jackknife is applied to a function of means, it gives the same answer as the delta method: